PROPAGATION OF VIBRATIONS IN A NONLINEAR DISSIPATIVE MEDIUM

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The present paper concerns the propagation of harmonic vibrations in a semi-infinite rod with nonlinear properties. The problem is solved on the basis of the harmonic linearization method.

The problem of the propagation of vibrations in nonlinear media is of great practical interest. One encounters it both in nonlinear acoustics [1] and in nonlinear optics [2]. Further the important effect of internal friction in materials is generally describable by means of nonlinear equations [3]. In describing the dynamic properties of complex structures it is possible in many cases to approximate the structure by a continuous medium. The properties of this medium often turn out to be nonlinear. In both of these cases it is the defining equations (i.e. the equations relating the stresses to the strains) which are nonlinear. This range of problems also includes those concerning vibrations in various classical nonlinear media (e.g., elastic-plastic, rigid-plastic, elastic-viscous-plastic media, reinforced media, etc.).

Problems on nonsteady-state perturbations in classical nonlinear media have been attracting the attention of numerous researchers over the past several years (see [4 to 6] et al.). However, to the author's knowledge vibrations have not been dealt with thus far.

We shall solve the problem of vibrations in the simplest case of a semi-infinite rod for certain types of nonlinearities. The solution will be approximate, giving a clear picture of the vibration field.

1. Let us consider longitudinal vibrations in a homogeneous semi-infinite rod $x > 0$. The rod dynamics equation is

$$\partial Q / \partial x - m \, \partial^2 u / \partial t^2 = 0 \tag{1.1}$$

Here u is the displacement of the rod cross section along the x-axis, Q is the tensile force in this cross section, t is time, and m is the linear mass of the rod, which is constant by hypothesis. We assume that strain in the rod is small and conforms to Expression

$$\varepsilon = \partial u / \partial x \tag{1.2}$$

which does not contain nonlinear terms.

We further assume that the defining equation is of the form

$$Q = Q\ (\varepsilon,\ \partial \varepsilon / \partial t) \tag{1.3}$$

so that the tensile force in any rod cross section depends solely on the strain and its rate of change in the same cross section. For simplicity we assume that the function Q is an odd function of its arguments.

As the boundary condition for $x = 0$ we take one of the following:

$$u = A_0 \cos \omega t, \qquad Q = B_0 \cos \omega t \tag{1.4}$$

The variables u and ε must tend to zero as $x \to \infty$.

2. One of the simplest and most effective approximate methods of nonlinear vibration theory is that of harmonic linearization. This method has been widely used in automatic control theory and in the theory of vibration absorbing systems for the analysis of nonlinear sys-

tems with a finite number of degrees of freedom [7 and 8]. An essential feature of this method is the simplicity and exceptional clarity of the physical results which it yields. We shall employ it to analyze the simplest nonlinear systems with distributed parameters.

In accordance with the harmonic linearization method we assume that the strain ε in each cross section varies with time according to a nearly harmonic law,

$$\varepsilon = a(x)\cos[\omega t - \varphi(x)] \tag{2.1}$$

Here a is the strain amplitude and φ is its phase for the cross section with the coordinate x.

Next, we approximate the nonlinear function $Q(\varepsilon, \partial\varepsilon/\partial t)$ in (1.3) by the linear function

$$Q(\varepsilon, \partial\varepsilon/\partial t) \approx q\varepsilon + (r/\omega)\partial\varepsilon/\partial t \tag{2.2}$$

The coefficients q and r in this expression are independent of time and are chosen from the requirement that the amplitude and phase of the harmonic of frequency ω in the right and left-hand sides of Eq. (2.2) be equal for harmonic motion (2.1). It has already been shown [7 and 8] that the q and r which satisfy this requirement are as follows: (2.3)

$$q = \frac{1}{\pi a}\int_0^{2\pi} Q(a\cos\psi - a\omega\sin\psi)\cos\psi\, d\psi, \qquad r = -\frac{1}{\pi a}\int_0^{2\pi} Q(a\cos\psi - a\omega\sin\psi)\sin\psi\, d\psi$$

As a result of the above linearization, nonlinear Eq. (1.3) has been replaced approximately by linear Expression (2.2). Introducing the latter into (1.1) and taking account of (1.2), we obtain

$$\frac{\partial}{\partial x}\left(q\frac{\partial u}{\partial x} + \frac{r}{\omega}\frac{\partial^2 u}{\partial x\,\partial t}\right) - m\frac{\partial^2 u}{\partial t^2} = 0 \tag{2.4}$$

This equation coincides in form with the equation of longitudinal vibrations of a viscous-elastic rod. The difference lies in the fact that in this case q and r depend on the strain amplitude a, i.e. they are not known in advance. Since q and r depend on the strain amplitude, it is expedient to differentiate Eq. (2.4) with respect to x and to introduce the strain ε as an unknown,

$$\frac{\partial^2}{\partial x^2}\left(q\varepsilon + \frac{r}{\omega}\frac{\partial\varepsilon}{\partial t}\right) - m\frac{\partial^2\varepsilon}{\partial t^2} = 0 \tag{2.5}$$

Let us substitute trial solution (2.1) into Eq. (2.5). Setting the coefficients of the cosines and sines equal to zero, we obtain two equations for determining the unknowns a and φ,

$$\frac{d^2(aq)}{dx^2} - (aq)\left(\frac{d\varphi}{dx}\right)^2 + 2\frac{d(ar)}{dx}\frac{d\varphi}{dx} + (ar)\frac{d^2\varphi}{dx^2} + m\omega^2 a = 0 \tag{2.6}$$

$$\frac{d^2(ar)}{dx^2} - (ar)\left(\frac{d\varphi}{dx}\right)^2 - 2\frac{d(aq)}{dx}\frac{d\varphi}{dx} - (aq)\frac{d^2\varphi}{dx^2} = 0 \tag{2.7}$$

Let us note the fact that not every solution of system (2.6), (2.7) constitutes a solution of the above problem. From the conditions at infinity it follows that the amplitude a must decrease with increasing x. This means that

$$a \geqslant 0, \qquad da/dx \leqslant 0 \tag{2.8}$$

Clearly, the solution of system (2.6), (2.7) with arbitrary q and r cannot be constructed in closed form even under limitations (2.8). However, such a solution can be constructed for certain particular q and r.

Thus, we have constructed equations for determining the strain. The displacement can be readily found by integrating over x in Expression (1.2).

Let us consider just what we have achieved by the formal application of the harmonic linearization method. Our starting point was system (1.1), (1.2), (1.3) equivalent to a single nonlinear partial differential equation. By employing harmonic linearization we obtained system (2.6), (2.7) of two ordinary nonlinear equations. Thus, one nonlinear problem has been reduced to another problem which is also nonlinear. The latter problem is definitely simpler than the original one, however.

3. Let nonlinear relation (1.3) be of the form

$$Q = G\,\mathrm{sign}\,\varepsilon + H\,\mathrm{sign}\,(\partial\varepsilon/\partial t) \qquad (G, H = \mathrm{const}) \tag{3.1}$$

Eq. (3.1) is the defining equation for a "material" with relay elastic and dissipative characteristics. It is easy to see that for $G = 0$ Eq. (3.1) refers to a rigid-plastic material.

Computing the linearization coefficients from Formulas (2.3), we obtain

$$q = g / a, \qquad r = h / a, \qquad g = 4G / \pi, \qquad h = 4H / \pi \tag{3.2}$$

Substituting them into Eqs. (2.6) and (2.7), we have

$$-g\left(\frac{d\varphi}{dx}\right)^2 + h\frac{d^2\varphi}{dx^2} + m\omega^2 a = 0, \qquad -h\left(\frac{d\varphi}{dx}\right)^2 - g\frac{d^2\varphi}{dx^2} = 0 \tag{3.3}$$

Integration of the second Eq. yields the Expression

$$\varphi = \varphi_0 + g / h \ln (1 + \gamma x) \tag{3.4}$$

where φ_0 and γ are integration constants. Substituting (3.4) into the first Eq. of (3.3), we obtain the following Expression for the amplitude:

$$a = \frac{g(g^2 + h^2)}{m\omega^2 h^2} \frac{\gamma^2}{(1 + \gamma x)^2} \tag{3.5}$$

We can find the strain ε by substituting (3.5), (3.4) into (2.1), and the displacement u by integrating ε,

$$\varepsilon = \frac{g(g^2 + h^2)}{m\omega^2 h^2} \operatorname{Re} \frac{\gamma^2 e^{i\omega t - i\varphi_0}}{(1 + \gamma x)^{2 + ig/h}} \tag{3.6}$$

$$u = \frac{g(g^2 + h^2)}{m\omega^2 h^2} \operatorname{Re} \frac{\gamma e^{i\omega t - i\varphi_1}}{(1 + ig/h)(1 + \gamma x)^{1 + ig/h}} \tag{3.7}$$

If the displacement varies according to law (1.4) in the cross section $x = 0$, then the constants γ and φ_0 must be chosen as follows:

$$\gamma = \frac{m\omega^2 A_0 h}{g\sqrt{g^2 + h^2}}, \qquad \varphi_0 = -\operatorname{arc\,tg} \frac{g}{h} \tag{3.8}$$

Introducing these quantities into (3.7), we finally obtain

$$u = \frac{A_0}{1 + \gamma x} \cos\left[\omega t - \frac{g}{h} \ln (1 + \gamma x)\right] \tag{3.9}$$

We note the fact that the vibration damping conditions as $x \to \infty$ (2.8) are fulfilled automatically.

Expression (3.9) implies that the phase velocity of the waves in the rod

$$v = \frac{\omega h}{g\gamma}(1 + \gamma x) \tag{3.10}$$

increases with distance from the vibration source. The vibration amplitude $A(x)$ diminishes according to the law

$$A(x) = A_0 \left(1 + \frac{m\omega^2 A_0 h x}{g\sqrt{g^2 + h^2}}\right)^{-1} \tag{3.11}$$

This readily yields the inequality

$$A(x) < \frac{g\sqrt{g^2 + h^2}}{m\omega^2 h x} \tag{3.12}$$

The vibrator oscillation amplitude A_0 does not appear in this expression, so that the vibrational level cannot be raised above a certain limit (which depends on the properties of the rod "material" and on the value of x) by any increase in the vibrator oscillation amplitude for any fixed x.

Let us consider the case of a rigid-plastic material. As noted above, this requires that we set $G = 0$. Taking the limit in solution (3.6), (3.9), we have

$$x = 0, \quad u = A_0 \cos\omega t, \quad a \to \infty; \qquad x > 0, \quad u = 0, \quad a = 0 \tag{3.13}$$

Our conclusion is as follows: In a rod of rigid-plastic material the vibrations are localized in the cross section $x = 0$ in which the vibrator lies.

4. Let the defining equation be of the form

$$Q = c\varepsilon + H \operatorname{sign} (\partial\varepsilon / \partial t) \tag{4.1}$$

It is clear that this expression is the strain law for a rigid-plastic material with linear reinforcement with allowance for the Bauschinger effect.

The linearization coefficients are

$$q = c, \qquad r = h / a, \qquad h = 4H / \pi \tag{4.2}$$

Eqs. (2.6) and (2.7) then become

$$c\frac{d^2a}{dx^2} - ca\left(\frac{d\varphi}{dx}\right)^2 + h\frac{d^2\varphi}{dx^2} + m\omega^2 a = 0 \tag{4.3}$$

$$h\left(\frac{d\varphi}{dx}\right)^2 + 2c\frac{da}{dx}\frac{d\varphi}{dx} + ca\frac{d^2\varphi}{dx^2} = 0 \tag{4.4}$$

Our attempts to solve this system of equations with its four integration constants proved unsuccessful. We were able, however, to find a solution containing two integration constants and satisfying conditions (2.8). This solution is

$$a = a_0 - \beta x, \qquad \varphi = \varphi_0 + \alpha x \tag{4.5}$$

Its direct substitution into system (4.3), (4.4) shows that the constants α and β must have the following values:

$$\alpha = (m\omega^2 / c)^{1/2}, \quad \beta = \alpha h / 2c \tag{4.6}$$

The constants a_0 and φ_0 are thus far arbitrary and are determined conventionally for $x = 0$. A solution of the form (4.5) is valid only for those x for which $a > 0$, i.e. for

$$0 \leqslant x < x_*, \qquad x_* = a_0 / \beta \tag{4.7}$$

For $x > x_*$ we assume that

$$a = 0, \qquad \varphi = \varphi_* = \varphi_0 + \alpha x_* \tag{4.8}$$

A direct check shows that this solution satisfies system (4.3), (4.4). In addition, it is continuously conjugate to solution (4.5). Substituting the resulting expressions for the amplitude and phase into (2.1), we obtain the following expression for the strain:

$$\varepsilon = (a_0 - \beta x)\cos(\omega t - \varphi_0 - \alpha x) \quad (0 < x < x_*), \qquad \varepsilon = 0 \quad (x > x_*) \tag{4.9}$$

Let the boundary condition for $x = 0$ be the second condition of (1.4). Satisfying this equation by means of the linearized expression for the force Q, we obtain the following values for the integration constants:

$$a_0 = \sqrt{B_0^2 - h^2} / c, \qquad \varphi_0 = \operatorname{arc\,tg}(h / a_0 c) \tag{4.10}$$

Hence we see that the solution constructed above has meaning only when the amplitude of the force applied in the cross section $x = 0$ is larger than h. From the physical standpoint this means that the force B_0 must exceed the yield stress H in (4.1). If the force B_0 is smaller than H, then there is no motion at all in the rod.

The displacement u in the problem under discussion can be found by integrating strain expression (4.9). The result is as follows:

$$u = -\frac{a_0 - \beta x}{\alpha}\sin(\omega t - \varphi_0 - \alpha x) +$$

$$+ \frac{h}{2\alpha c}[\cos(\omega t - \varphi_0 - \alpha x) - \cos(\omega t - \varphi_0 - \alpha x_*)], \qquad 0 < x < x_*$$

$$u = 0, \qquad x > x_* \tag{4.11}$$

Hence we see that the displacement in the initial zone of the rod constitutes a superposition of a travelling and a standing wave. There is no motion for $x > x_*$. The coordinate x_* of the motion zone boundary can be found by substituting Expressions (4.10) and (4.6) into (4.7),

$$x_* = \frac{2\sqrt{B_0^2 - h^2}}{h}\left(\frac{c}{m\omega^2}\right)^{1/2} \tag{4.12}$$

5. Let the rod be made of an elastic-plastic material with a linear reinforcement law. The behavior of the rod under developed plastic strains is described by the system

$$Q = c\varepsilon + c_1(\varepsilon - \varepsilon_1), \qquad c_1(\varepsilon - \varepsilon_1) = H \operatorname{sign}(\partial\varepsilon_1 / \partial t) \tag{5.1}$$

where ε_1 is the plastic strain (Fig. 1).

The function sign $(\partial\varepsilon_1/\partial t)$ is equal to $+1$ for a positive rate of plastic strain and to -1 for a negative rate. We further assume that with the rate of plastic strain equal to zero, sign takes on a value in the range $(-1, +1)$ which is dictated by the second Eq. of (5.1). With the sign function so defined, system (5.1) can be used to describe rod loading and unloading processes both with and without plastic strains and with allowance for the Bauschinger effect (Fig. 2).

Analysis of Eqs. (5.1) leads to the following conclusions. In the absence of plastic strains the rigidity of the rod is equal to $c + c_1$. Plastic strain begins when the distending force Q attains the value $H(c + c_1)/c_1$. The reinforcement of the rod is determined by the parameter c.

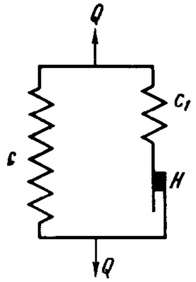

Fig. 1

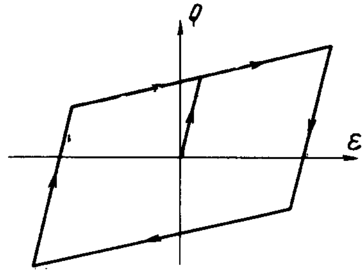

Fig. 2

Let us consider the boundary conditions as $x \to \infty$. An elastic-plastic rod can move in such a way that plastic strains do not arise. In this case the vibrations need not tend to zero as $x \to \infty$ as assumed above. Hence, the Sommerfeld radiation condition must be taken as the boundary condition. This condition states that the waves go out to infinity and do not return.

In accordance with the method of harmonic linearization we set

$$\varepsilon_1 = b \cos(\omega t - \psi) \tag{5.2}$$

where b and ψ are the amplitude and phase of the plastic strain.

Further, we approximate the nonlinear function in (5.1) by a linear function [7 and 8],

$$H \operatorname{sign}\left(\frac{\partial \varepsilon_1}{\partial t}\right) \approx \frac{h}{b\omega}\frac{\partial \varepsilon_1}{\partial t}, \qquad h = \frac{4H}{\pi}$$

Let us substitute this approximation into (5.1) and find expressions for ε and Q in terms of ε_1

$$\varepsilon = \varepsilon_1 + \frac{h}{c_1 b\omega}\frac{\partial \varepsilon_1}{\partial t}, \qquad Q = c\varepsilon_1 + \left(\frac{c}{c_1} + 1\right)\frac{h}{b\omega}\frac{\partial \varepsilon_1}{\partial t} \tag{5.3}$$

Differentiating (1.1) with respect to x and making use of Formulas (1.2) and (5.3), we obtain the following Eq. for determining the plastic strain:

$$\frac{\partial^2}{\partial x^2}\left[c\varepsilon_1 + \left(\frac{c}{c_1} + 1\right)\frac{h}{b\omega}\frac{\partial \varepsilon_1}{\partial t}\right] - m\frac{\partial^2}{\partial t^2}\left(\varepsilon_1 + \frac{h}{b\omega}\frac{\partial \varepsilon_1}{\partial t}\right) = 0 \tag{5.4}$$

Finally, let us set the Expression (5.2) for ε_1 into (5.4) and collect the coefficients of the sines and cosines. This yields a system of two ordinary equations for determining the amplitude and phase of plastic strain,

$$c\left[\frac{d^2 b}{dx^2} - \left(\frac{d\psi}{dx}\right)^2 b\right] + h\left(\frac{c}{c_1} + 1\right)\frac{d^2\psi}{\partial x^2} + m\omega^2 b = 0 \tag{5.5}$$

$$-c\left[b\frac{d^2\psi}{\partial x^2} + 2\frac{db}{dx}\frac{d\psi}{dx}\right] - \left(\frac{c}{c_1} + 1\right)h\left(\frac{d\psi}{dx}\right)^2 + \frac{m\omega^2 h}{c_1} = 0 \tag{5.6}$$

The above system of equations has the solution

$$b = b_0 - \beta x, \qquad \psi = \psi_0 + \alpha x \tag{5.7}$$

where b_0 and ψ_0 are integration constants and α and β are easy to find by substituting Expressions (5.7) into system (5.5), (5.6),

$$\alpha = (m\omega^2 / c)^{1/2}, \qquad \beta = h\alpha / 2c \tag{5.8}$$

Solution (5.7) has meaning only for that portion of the rod where the amplitude b is positive, i.e. for x satisfying the inequality

$$0 < x < x_*, \qquad x_* = b_0 / \beta \tag{5.9}$$

For $x > x_*$ we must take

$$\left(\psi_* = \psi_0 + \alpha x_*,\ \gamma = \left(\frac{m\omega^2}{c + c_1}\right)^{1/2}\right) \qquad b = 0,\ \psi = \psi_* + \gamma(x - x_*) \tag{5.10}$$

Direct substitution of solution (5.10) into system (5.5), (5.6) shows that it satisfies this system. In addition, it is continuously conjugate to solution (5.7) in the range $0 < x < x_*$. In fact, the amplitude and phase of plastic strain ε_1 are continuous at $x = x_*$. But by virtue of relations (5.3) this means that the total strain and stress in the rod are also continuous.

Thus, for $x > x_*$ the plastic strain equals zero. The solution just constructed satisfies the boundary condition at infinity, since $d\Psi/dx > 0$, $b = 0$ as $x \to \infty$.

6. Let us determine the integration constants and find the basic parameters of the problem, i.e. the total strain ε and the displacement u. This is most readily accomplished by writing solution (5.2) in complex form,

$$\varepsilon_1 = be^{i\omega t - i\psi} \tag{6.1}$$

Conversion to the complex form is possible because relations (5.3) are linear with respect to ε_1. Of course, only the real parts of (6.1) and all subsequent expressions have physical meaning. Substituting (6.1) into (5.3), we obtain

$$Q = \left[cb + ih\left(\frac{c + c_1}{c_1}\right)\right] e^{i\omega t - i\psi}, \qquad \varepsilon = \left(b + \frac{ih}{c_1}\right) e^{i\omega t - i\psi} \tag{6.2}$$

Making use of relation (6.2) and boundary condition (2.4) for $x = 0$, we readily obtain the initial values of the amplitude and phase of plastic strain,

$$b_0 = \frac{1}{c}\left[B_0^2 - h^2\left(\frac{c + c_1}{c_1}\right)^2\right]^{1/2}, \qquad \psi_0 = \operatorname{arctg}\frac{h\,(c + c_1)}{cc_1 b_0} \tag{6.3}$$

From this we see that the solution constructed has meaning only if the amplitude of the constraining force satisfies the inequality

$$B_0 > h\,(c + c_1)\,/\,c_1 \tag{6.4}$$

This is the condition which must be fulfilled in order for plastic strains to occur in the rod. It is approximate and differs from the exact condition only by the unimportant factor $4/\pi$ which enters into h.

From Formula (6.2) we see that the strain amplitude is given by

$$a = (b^2 + h^2\,/\,c_1^2)^{1/2} \tag{6.5}$$

or, with allowance for the explicit expression for b,

$$a = [(b_0 - \beta x)^2 + h^2\,/\,c_1^2]^{1/2} \quad (0 < x < x_*), \qquad a = h\,/\,c_1 \quad (x > x_*) \tag{6.6}$$

This implies that in the initial portion of the rod the strain amplitude diminishes from its maximum value at the cross section $x = 0$ to the value h/c_1 and then remains constant and essentially equal to the maximum elastic strain.

Let us find the displacement u. To do this we substitute into (6.2) the explicit expressions for b and ψ from Formulas (5.7), (5.10), and then integrate over x taking into account the requirement that u must be continuous at $x = x_*$ as well as the radiation condition as $x \to \infty$. The result is as follows:

$$0 < x < x_*, \qquad u = \left(i\frac{b}{\alpha} - \frac{h}{c_1\alpha} - \frac{\beta}{\alpha^2}\right) e^{i\omega t - i\psi} + \left[\frac{h}{c_1}\left(\frac{1}{\alpha} - \frac{1}{\gamma}\right) + \frac{\beta}{\alpha^2}\right] e^{i\omega t - i\psi_*}$$

$$x > x_*, \qquad u = -\frac{h}{c_1\gamma}\, e^{i\omega t - i\psi} \tag{6.7}$$

The quantities appearing in this expression have already been defined. From Expression (6.7) we see that for $x > x_*$ the constant-amplitude wave goes out to infinity. For $0 < x < x_*$ the vibrations constitute a superposition of travelling and standing waves, where the travelling wave decreases in amplitude with distance from the vibration source.

7. Let us consider some conclusions. The cross section $x = x_*$ is the boundary between that portion of the rod in which plastic strains occur and the portion in which the strain is purely elastic. An explicit expression for x_* can be obtained by substituting into (5.9) the expressions for b_0 from (6.3) and for β, α from (5.8),

$$x_* = \frac{2}{h}\left[B^2 - h^2\left(\frac{c + c_1}{c_1}\right)^2\right]^{1/2}\left(\frac{c}{m\omega^2}\right)^{1/2} \tag{7.1}$$

Hence we see that the zone of plastic strains is larger the larger the reinforcement of the rod material, the smaller its linear mass, and the lower the vibration frequency. Specifically, if the rod material is ideally plastic ($c = 0$), then (7.1) implies that $x_* = 0$, i.e. the zone of plastic strains is localized in the cross section containing the vibration source.

We note the fact that the plastic strain zone is at the same time the zone where the vibration intensity is highest, i.e. the zone characterized by large displacements. In the elastic range ($x > x_*$) the vibration amplitude is minimal, and, as we see from (6.7), does not depend on the vibrator strength, but rather on the properties of the rod material and (as is evident from the expression for γ) on the frequency: the higher the frequency, the smaller the vibration amplitude.

Finally let us consider the instantaneous photograph of the rod. The plastic strain zone $0 - x_*$ constitutes a sequence of plastic strain segments of differing sign. The number of these segments is equal to the number of extrema in ε_1. This number can be readily determined from Formula (5.2),

$$n = (\psi_* - \psi_0) / \pi \tag{7.2}$$

Substituting in the required quantities, we obtain

$$n = \frac{2}{\pi h}\left[B^2 - h^2\left(\frac{c + c_1}{c_1}\right)^2\right]^{1/2} \tag{7.3}$$

If the perturbation amplitude B exceeds the yield stress considerably and if the reinforcement of the material is not too large, the second term in square brackets is negligibly small. Leaving out this term, we have

$$n = 2B / \pi h \tag{7.4}$$

Hence we see that the number of segments can be substantial. This in turn renders difficult the construction of an exact solution, since in this case we are dealing with a sequence of loading and "unloading" waves [4 and 5].

BIBLIOGRAPHY

1. Zarembo, L.K. and Krasil'nikov, V.A., Introduction to Nonlinear Acoustics, Moscow, Izd. "Nauka", 1966.
2. Bloembergen, N., Nonlinear Optics, Moscow, Izd. "Mir", 1966.
3. Panovko, Ia.G., Internal Friction in the Vibration of Elastic Systems, Moscow, Fizmatgiz, 1960.
4. Shapiro, G.S., Longitudinal vibrations of a rod. PMM Vol. 10, No. 5/6, 1946.
5. Rakhmatulin, Kh.A. and Shapiro, G.S., On the propagation of plane elastic-plastic waves. PMM Vol. 12, No. 4, 1948.
6. Barenblatt, G.I. and Ishlinskii, A.Iu., On the impact of a visco-plastic bar on a rigid obstacle. PMM Vol. 26, No. 3, 1962.
7. Popov, E.P. and Pal'tov, I.P., Approximate Methods of Investigating Nonlinear Automatic Systems, Moscow, Fizmatgiz, 1960.
8. Kolovskii, M.V., Nonlinear Theory of Vibration Proofing Systems, Moscow, Izd. "Nauka", 1966.

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